A Problem Connected with the Schott Term in Classical Electrodynamics

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It is shown that the well-known "4/3-problem" of the old Abraham-Lorentz model for the extended electron is still present in Rohrlich's redefinition of the electromagnetic four-momentum, if the latter is applied to an accelerated electron. The notorious factor 4/3 emerges now in connection with the Schott term. Since this term is an indispensable constituent of the Lorentz-Dirac equation of motion for the radiating electron, this equation appears to be not completely reliable.

I. The Problem

It is sometimes said that there are some perpetual problems in physics, which never have been solved satisfactorily but have on the other hand not prevented the overwhelming success of modern physics. In the present paper we are concerned with a problem of this sort: it is the old "4/3-problem" *.

This difficulty emerged for the first time in the classical electron models of Abraham and Lorentz, when the electromagnetic energy and momentum of a uniformly moving electron was to be calculated. In the non-relativistic approximation, Lorentz found on the basis of his "contraction hypothesis" the electromagnetic self-energy E of a spheroidal electron as

$$E = \frac{Z^2}{2\,r_{\mathrm{c}}} + \frac{1}{2}\,\frac{\boldsymbol{v}^2}{c^2}\left(\frac{5}{3}\,\frac{Z^2}{2\,r_{\mathrm{c}}}\right), \qquad (\mathrm{I},1)$$

where r_{c} is the radius of the sphere (in the rest system) being assumed to be uniformly charged. Clearly, this self-energy does not constitute, even in the non-relativistic approximation, the time component of a momentum four-vector. This was the first difficulty of the Lorentz model. The second difficulty arose, when the (linear) momentum of the uniformly moving electron was to be computed. The result was

$$\boldsymbol{P} = \frac{4}{3}\,\frac{Z^2}{2\,r_{\mathrm{c}}}\,\frac{\boldsymbol{v}}{c}. \qquad (\mathrm{I},2)$$

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* As an overall reference of the relevant literature in this respect, the book of Sexl and Urbantke [1] or Rohrlich's paper [2] are recommendable.

The inertial mass of the electron exhibits an additional factor 4/3 compared to the rest self-energy (I, 1). This is the "4/3-problem".

Rohrlich [2-5] seems to be the man, who has worked most thoroughly on this problem, and he has shown a way out of the trouble by defining the energy-momentum four-vector not as an integral over the "now-plane" (σ_{r}) of an observer at rest:

$$P^{\mu}_{(\sigma_{\mathrm{r}})} = \frac{1}{c}\int\limits_{(\sigma_{\mathrm{r}})} T^{\mu}{}_{\nu}\,\mathrm{d}^3\sigma_{\mathrm{r}}{}^{\nu} = \frac{1}{c}\int T^{\mu}{}_0\,\mathrm{d}^3\boldsymbol{x}, \qquad (\mathrm{I},3)$$

as Abraham and Lorentz have done, but in defining the electron's four-momentum by a hyperplane integration over the moving particle's "now-plane" ($\sigma_{\perp}$):

$$P^{\mu}_{(\sigma_{\perp})} = \frac{1}{c}\int\limits_{(\sigma_{\perp})} T^{\mu}{}_{\nu}\,\mathrm{d}^3\sigma_{\perp}{}^{\nu}. \qquad (\mathrm{I},4)$$

The result of this definition is a correctly transforming momentum four-vector

$$P^{\mu}_{(\sigma_{\perp})} = \frac{1}{c}\,\frac{Z^2}{2\,r_{\mathrm{c}}}\,u^{\mu}, \qquad (\mathrm{I},5)$$

where $\{u^{\mu}\} = \{\gamma;(\boldsymbol{v}/c)\,\gamma\}$ and $\gamma = (1-\boldsymbol{v}^2/c^2)^{-1/2}$.

The advantage of Rohrlich's method was, that the transformation problem has been decoupled from the stability problem: though no additional cohesive forces (Poincaré stresses) had to be introduced in order to prevent the charge from exploding, the correct transformation behaviour of energy and momentum was achieved.

So far, so good. But, up to now, all considerations were directed towards the uniformly moving electron. Peierls [6] has recently pointed out that if the new definition (I, 4) should be really meaningful, it must work also in the case of an arbitrarily

accelerated electron. This problem was first solved by Teitelboim[7] for a point electron and later on by our own paper[8] for an arbitrarily extended electron. Teitelboim's result, to which the result for the extended particle[8] reduces in the limit of vanishing size ($r_c \Rightarrow \varepsilon \to 0$), was **

$$P^\mu = \frac{1}{c}\frac{Z^2}{2\,\varepsilon}u^\mu - \frac{2}{3}\frac{Z^2}{c}\dot{u}^\mu \,. \qquad (\mathrm{I},6)$$

Here the Schott term (second term on the right) emerged as an additional four-momentum of the bound velocity fields dragged along by the electron. The 4/3-factor is missing in the first term now as before, but the factor 2/3 in connection with the Schott term suggests that the 4/3-difficulty is now attached to this term.

The present paper claims to prove this suggestion by interpreting the Schott term as the relative four-momentum of the Coulomb field (emitted at earlier times but observed instantaneously in the momentary rest frame of the particle) with respect to a comoving observer, who is at rest in each of the momentary rest frames. Since the Schott term is unavoidably incorporated in the Lorentz-Dirac equation of motion for the radiating electron

$$m\,c^2\,\dot{u}^\lambda = K^\lambda + \tfrac{2}{3}Z^2\left\{\ddot{u}^\lambda + (\dot{u}\,\dot{u})u^\lambda\right\}, \qquad (\mathrm{I},7)$$

this equation suffers also from the notorious "4/3-illness" and therefore does not seem to be especially reliable.

In the following, we proceed in four steps: First, we investigate the "4/3-problem" within the framework of the retarded geometry (Section II). Then the surface independence of the relevant hyperplane integrations is analyzed (Section III). After this is done, one obtains an expression for the hyperplane integral, where the plane used does not coincide with the resting observer's "now-plane" nor with that of the uniformly moving particle. On this level of approach, a splitting of the four-momentum is performable, which introduces the notion of a relative-momentum vector (Section IV). This vector is then shown to be the generalization of the Schott term in the case of an extended electron and has the Schott term as a point limit (Section V). But it is exactly this relative four-momentum, which exhibits the old "4/3-difficulty".

** We are concerned here only with the bound four-momentum. The radiated four-momentum is consequently omitted.

II. The 4/3-Difficulty within the Framework of the Retarded Geometry

In this section we shall first re-examine the old method (I, 3) of defining the electron's four-momentum P^μ as an integral over the instantaneous "now-plane" (σ_r) of the observer at rest.

Of course, the energy-momentum tensor $T^{\mu\nu}$ is here constructed according to the usual prescription[5]

$$T^{\mu\nu} = -\frac{1}{4\,\pi}\left\{F^\mu{}_\lambda F^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right\}, \qquad (\mathrm{II},1)$$

where the field strengths $F^{\mu\lambda}$ are obtainable from the Liénard-Wiechert potentials

$$A^\lambda{}_{(x)} = Z\,u^\lambda{}_{(s_{ret})}/R \qquad (\mathrm{II},2)$$

by

$$F^{\mu\nu} = A^{\mu/\nu} - A^{\nu/\mu}\,. \qquad (\mathrm{II},3)$$

Since we are dealing in this section with a uniformly moving electron, one finds easily by Synge's[9] method of retarded differentiation

$$F^{\mu\nu} = (Z/R^2)\,(u^\mu v^\nu - u^\nu v^\mu)\,. \qquad (\mathrm{II},4)$$

The retarded distance R of the point $\{x^\lambda\}$ from the world line $x^\lambda = z^\lambda{}_{(s)}$ is given by

$$R = (x^\lambda - z^\lambda{}_{(s)})\cdot u_{\lambda(s)}$$

and $\{v^\lambda\}$ is an angular-dependent space-like unit vector in the orthogonal plane ($\sigma_\perp$) to the world line (for further definitions of the notation see the earlier paper[8]).
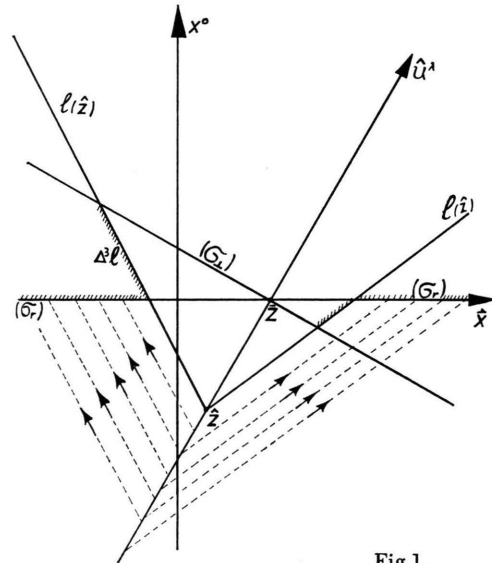

Fig 1.

Now consider a uniformly moving (velocity: $\hat{u}^\lambda \equiv \mathrm{d}z^\lambda/\mathrm{d}s$) electron. An observer being at rest wants to determine the electromagnetic four-momentum of the electron in the space-time point $\{\bar{z}^\lambda\}$ on the world line of the particle. Since the observer's world line is parallel to the x^0-axis (see Fig. 1), he would naively try to integrate the energy-momentum density over his own "now-plane" (σ_r) given by the constraint $x^0 = $ const. But since the fields (II, 4) are singular in the point $\{\bar{z}^\lambda\}$ of his "now-plane", the observer would like to cut out a hole around the singular point $\{\bar{z}^\lambda\}$, the interior of which is then excluded from the integration. Clearly, an integration of this sort

$$P^\mu_{(\sigma_\mathrm{r}')} = \frac{1}{c} \int\limits_{(\sigma_\mathrm{r}')} T^\mu_{\ \nu} \, \mathrm{d}^3\sigma_\mathrm{r} \qquad (\mathrm{II}, 5)$$

[the prime in (σ_r') indicates the exclusion of the hole] sums up the whole history of the particle, because the electromagnetic field on (σ_r) stems from the past positions of the electron (dotted lines in Figure 1). It seems most natural to determine the hole around $\{\bar{z}^\lambda\}$ in such a way, that a last signal emitted by the electron in $\{\hat{z}^\lambda\}$ with

$$(\bar{z}^\lambda - \hat{z}^\lambda)\,(\bar{z}_\lambda - \hat{z}_\lambda) = \Delta s^2 = \text{const}$$

separates the interior of the hole from the outside region on (σ_r). The boundary $(S_\mathrm{r}$, say) of the hole is thereby the intersection of the light cone $l_{(\hat{z})}$ with the plane (σ_r). The parameter Δs is held fixed; one can assume it as the classical electron radius determined by

$$Z^2/2\,\Delta s = m\,c^2 , \qquad (\mathrm{II}, 6)$$

where m is the experimental mass of the electron. However, if one wants to have a point electron, one lets Δs tend to zero.

Each point $\{\xi^\lambda\}$ on the boundary (S_r), which is a two-dimensional sphere of radius ϱ

$$\varrho = (\bar{z}^\lambda - \hat{z}^\lambda)\,u_\lambda = \Delta s\,(u\,\hat{u}) = \Delta s \cdot \hat{u}^0 \quad (\mathrm{II}, 7)$$

(let $\{u^\lambda\} = \{1; 0, 0, 0\}$ be the four-velocity of the observer at rest), is satisfying the two equations

$$(\xi^\lambda - \hat{z}^\lambda)\,(\xi_\lambda - \hat{z}_\lambda) = 0 \qquad (\mathrm{II}, 8a)$$

$$u^\lambda(\xi_\lambda - \bar{z}_\lambda) = 0 , \qquad (\mathrm{II}, 8b)$$

because it is contained as well in the light cone $l_{(\hat{z})}$ as in (σ_r), which has normal $\{u^\lambda\} = \{1; 0, 0, 0\}$. Hence, one can write

$$\xi^\lambda = \hat{z}^\lambda + R_{\min}\,\hat{n}^\lambda , \qquad (\mathrm{II}, 9)$$

where

$$\hat{n}^\lambda = \hat{u}^\lambda + \hat{v}^\lambda \qquad (\mathrm{II}, 10)$$

and

$$R_{\min} = \varrho/(\hat{n}\,u) . \qquad (\mathrm{II}, 11)$$

After these geometrical preliminaries, one can perform the integration in (II, 5) quite conveniently by applying Gauß' integral theorem (observe $T^{\mu\nu}{}_{/\nu} \equiv 0$ off the world line) to the four-dimensional volume bound by $l_{(\hat{z})}$, (σ_r), and an infinitely distant surface (Σ). Clearly, the flux through (Σ) vanishes, because we are dealing with an electron in uniform motion (no radiation present), and so we arrive at the equality

$$\int\limits_{(\sigma_\mathrm{r}')} T^\mu_{\ \nu} \, \mathrm{d}^3\sigma_\mathrm{r}{}^\nu = \int\limits_{l'_{(\hat{z})}} T^\mu_{\ \nu} \, \mathrm{d}^3l^\nu_{(\hat{z})} , \qquad (\mathrm{II}, 12)$$

where the light cone surface element is [9]

$$\mathrm{d}^3l^\nu_{(\hat{z})} = \hat{n}^\nu\,R^2\,\mathrm{d}R\,\mathrm{d}\hat{\Omega} . \qquad (\mathrm{II}, 13)$$

(Compare the quite similar argumentation in the first paper [8] referring to the calculation of the bound four-momentum P_b^μ.) Therefore

$$P^\mu_{(\sigma_\mathrm{r}')} = \frac{1}{c} \int\limits_{l'_{(\hat{z})}} T^\mu_{\ \nu} \, \mathrm{d}^3l^\nu_{(\hat{z})} = \frac{1}{c}\,\frac{Z^2}{2\,\varrho}$$
$$\cdot \left\{ \tfrac{4}{3}\,\hat{u}^\mu(u\,\hat{u}) - \tfrac{1}{3}\,u^\mu \right\} . \qquad (\mathrm{II}, 14)$$

This result can be taken over directly from Reference [8]. Now we have

$$(u\,\hat{u}) = \hat{u}^0 =: \hat{\gamma} := \left\{1 - (\boldsymbol{v}_\mathrm{rel}/c)^2\right\}^{-1/2} , \quad (\mathrm{II}, 15)$$

where $\boldsymbol{v}_\mathrm{rel}$ is the ordinary three-velocity of the electron relative to the observer at rest. From here one easily calculates the energy (note $\varrho = \hat{\gamma} \cdot \Delta s$)

$$E_{(\sigma_\mathrm{r}')} = c\,P^0_{(\sigma_\mathrm{r}')} = \frac{Z^2}{2\,\Delta s}\,\frac{4\,\hat{\gamma}^2 - 1}{3\,\hat{\gamma}} = m\,c^2\,\frac{4\,\hat{\gamma}^2 - 1}{3\,\hat{\gamma}} . \qquad (\mathrm{II}, 16)$$

This is, of course, not the zero component of a momentum four-vector, and so we recover the old difficulty of Abraham and Lorentz mentioned in the introduction. Moreover, the present result does not agree with that of Lorentz for relativistic motion. But this is not astonishing, because we have not used the Lorentz-contracted shape of the hole. However, expanding (II, 16) in powers of $(\boldsymbol{v}_\mathrm{rel}/c)$, one finds

$$E_{(\sigma_\mathrm{r}')} = m\,c^2\left(1 + \frac{5}{3} \cdot \frac{\boldsymbol{v}_\mathrm{rel}^2}{2\,c^2} + \dots\right), \quad (\mathrm{II}, 17)$$

which means agreement with Lorentz's approximative formula (I, 1) up to second order in $(\boldsymbol{v}_\mathrm{rel}/c)$.

Now we take the space part of the four-vector formula (II, 14), and recover exactly the relativistic Lorentz result

$$P_{(\sigma_{r'})} = m\,c\,\frac{4}{3}\,\hat{\boldsymbol{u}} = \frac{4}{3} \cdot \frac{m\,\boldsymbol{v}_{\mathrm{rel}}}{\sqrt{1-(\boldsymbol{v}_{\mathrm{rel}}/c)^2}}\,. \qquad (\mathrm{II},18)$$

As Rohrlich's definition (II, 1) leads to

$$P^\mu = m\,c\,u^\mu = m\,c\,\hat{\gamma}\,\{1; \boldsymbol{v}_{\mathrm{rel}}/c\}\,, \qquad (\mathrm{II},19)$$

we see that one cannot remedy the 4/3-illness by means of the retarded geometry.

It is often assured in the literature, that the non-vanishing divergence of the energy-momentum tensor is the cause for this 4/3-difficulty. If this divergence would namely vanish, one could achieve by means of Gauß' theorem the equivalence of (I, 4) and (I, 3) in simply applying this theorem to a four-volume enclosed by the hypersurface (σ_r), by the orthogonal hyperplane $(\sigma_\perp)$ to the world line in $\{\bar{z}^\lambda\}$, and by an additional infinitely distant surface (Σ), which would however yield no contribution. But one should be somewhat cautious with such a statement, because we have integrated only over a space-time region, where the tensor divergence of $T^{\mu\nu}$ actually vanishes. The fault is not really the field singularity on the world line but rather the impossibility of constructing a hole around the singular point $\{\bar{z}^\lambda\}$ being contained in both planes (σ_r) and $(\sigma_\perp)$. If one does not insist on the use of hyperplanes but admits also curved surfaces, which fulfill only the requirement of intersecting one another in a definite hole around $(\{\bar{z}^\lambda\})$, then one has a certain independence of the surface integral under consideration even in the presence of a field singularity. We shall present a simple example of this sort in the next section.

III. Surface Independence of the Definition of the Four-Momentum

As was briefly indicated in the foregoing section, Rohrlich's definition (I, 4) yields a reasonable result [formula II, 19]. In following Rohrlich's idea, a hole had to be cut out of the plane $(\sigma_\perp)$ in order that the integral in (I, 4) were meaningful. We shall now construct a second surface $(\tilde{\sigma}$, say), which intersects $(\sigma_\perp)$ exactly in the boundary of this hole being cut out of $(\sigma_\perp)$. Let us call this boundary $(S_\perp)$; it can be thought to be the intersection of the light cone $l_{(\hat{z})}$ with vertex in $\{\hat{z}^\lambda\}$ with the ortho-gonal hyperplane $(\sigma_\perp)$ to the world line in the point $\{\bar{z}^\lambda\}$, where the old constraint $(\bar{z}-\hat{z})^2 = \Delta s^2$ has been applied again.

As a second surface $(\tilde{\sigma})$, which intersects $(\sigma_\perp)$ in the two-dimensional closed surface $(S_\perp)$, we can choose a combination of two segments: the first segment is (σ_r') out of (σ_r); the second segment is that part $(\Delta^3 l)$ out of $l_{(\hat{z})}$, which is limited by (S_r) and $(S_\perp)$. Observe, that (σ_r) intersects $l_{(\hat{z})}$ in (S_r), and that $(\sigma_\perp)$ intersects $l_{(\hat{z})}$ in $(S_\perp)$. The new surface $(\tilde{\sigma})$ is sketched with fringes in Figure 1.

Of course, we know from Gauß' integral theorem, applied to the four-volume enclosed by $(\sigma_\perp')$ and $(\tilde{\sigma})$, that the identity

$$\int\limits_{(\sigma_\perp')} T^\mu{}_\nu\,\mathrm{d}^3\sigma_{\perp}{}^\nu = \int\limits_{(\tilde{\sigma})} T^\mu{}_\nu\,\mathrm{d}^3\,\tilde{\sigma}^\nu \qquad (\mathrm{III},1)$$

must be valid. But we want to assure ourselves by explicit calculation: the left-hand side of (III, 1) is Rohrlich's result (II, 19) and the right-hand side splits up into two parts

$$\int\limits_{(\tilde{\sigma})} T^{\mu\nu}\,\mathrm{d}^3\tilde{\sigma}^\nu = \int\limits_{(\sigma_{r'})} T^\mu{}_\nu\,\mathrm{d}^3\sigma_r{}^\nu + \int\limits_{\Delta^3 l} T^\mu{}_\nu\,\mathrm{d}^3 l^\nu\,. \qquad (\mathrm{III},2)$$

Since the first member of the right-hand side of this equation is known from (II, 14)

$$\int\limits_{(\sigma_{r'})} T^\mu{}_\nu\,\mathrm{d}^3\sigma_r{}^\nu = (Z^2/2\,\varrho)\,\{\tfrac{4}{3}\,\hat{u}^\mu\,(u\,\hat{u}) - \tfrac{1}{3}\,u^\mu\}\,, \qquad (\mathrm{III},3)$$

we are only left with the second member (see Ref. [8]):

$$\begin{aligned}
\int\limits_{(\Delta^3 l)} T^\mu{}_\nu\,\mathrm{d}^3 l^\nu &= \int\limits_{R=\Delta s}^{R_{\min}} \frac{Z^2}{8\,\pi} \cdot \frac{\hat{n}^\mu}{R^4}\,R^2\,\mathrm{d}R\,\mathrm{d}\hat{\Omega} \\
&= \frac{Z^8}{8\,\pi}\int \mathrm{d}\hat{\Omega}\,\hat{n}^\mu\left(\frac{1}{\Delta s} - \frac{(\hat{n}\,u)}{\varrho}\right) \qquad (\mathrm{III},4) \\
&= \frac{Z^2}{2\,\Delta s}\,\hat{u}^\mu - \frac{Z^2}{2\,\varrho}\,\{\tfrac{4}{3}\,\hat{u}^\mu\,(u\,\hat{u}) - \tfrac{1}{3}\,u^\mu\}\,.
\end{aligned}$$

Inserting (III, 3) and (III, 4) into (III, 2) yields

$$\int\limits_{(\tilde{\sigma})} T^\mu{}_\nu\,\mathrm{d}^3\,\tilde{\sigma}^\nu = (Z^2/2\,\Delta s)\,\hat{u}^\mu = \int\limits_{(\sigma_\perp')} T^\mu{}_\nu\,\mathrm{d}^3\sigma_\perp{}^\nu\,, \qquad (\mathrm{III},5)$$

which completes the desired proof. Conversely, we conclude from the above considerations, that the difference of the two hyperplane integrals involved is just the flux through the the light-cone segment $(\Delta^3 l)$

$$\int\limits_{(\sigma_\perp')} T^\mu{}_\nu\,\mathrm{d}^3\sigma_\perp{}^\nu - \int\limits_{(\sigma_{r'})} T^\mu{}_\nu\,\mathrm{d}^3\sigma_r{}^\nu = \int\limits_{(\Delta^3 l)} T^\mu{}_\nu\,\mathrm{d}^3 l^\nu\,. \qquad (\mathrm{III},6)$$

Hence, this flux is responsible for the non-equality of Rohrlich's definition (I, 4) and the old definition (I, 3) of Abraham and Lorentz.

IV. The Four-Momentum of Internal Relative Motion

In this section, we first want to generalize the old definition $(I, 3)$ together with its result of integration $(II, 14)$. Clearly, we have written $(II, 14)$ in a far more general form as would have been necessary by the specialization $\{u^\lambda\} = \{1; 0, 0, 0\}$. Indeed, the result $(II, 14)$ is valid in the case, where $\{u^\lambda\}$ in its property of a unit normal vector to the plane of integration has the most general form of a (reduced) velocity four-vector. The plane of integration may then be considered as completely arbitrary and subjected only to the condition of being space-like and containing the point $\{\tilde{z}^\lambda\}$. If, for instance, this arbitrary plane coincides with $(\sigma_\perp)$, one has $\{u^\lambda\} = \{\hat{u}^\lambda\}$ and $(II, 14)$ reduces to Rohrlich's result $(II, 19)$, as it must be.

We can now consider the arbitrary new plane $(\sigma_n$, say) just introduced to differ very little * from the orthogonal plane $(\sigma_\perp)$. The new plane (σ_n) is the "now-plane" of an observer, who sees the electron moving very slowly relative to himself. The four-momentum $(II, 14)$, obtained by integration over (σ_n) with $\{u^\lambda\}$ now being the four-velocity of the observer in slow relative motion with respect to the electron, can be regarded as composed of two parts: the first part consists of the four-momentum, which the electron would have, if it were at rest relative to the moving observer just introduced, i. e., if the electron would move with four-velocity $\{u^\lambda\}$. Since the observer sees in his "now-plane" a Coulomb field with a sphere of radius ϱ cut out, the first part is assumed to be

$$P_{\mathrm{I}}{}^\mu = \frac{1}{c}\,\frac{Z^2}{2\,\varrho}\,u^\mu . \qquad (IV, 1)$$

But, of course, the observer does not see a static Coulomb field, because the electron moves with velocity $\{\hat{u}^\lambda\} \neq \{u^\lambda\}$. Therefore, we must add to $(IV, 1)$ a second part, namely the four-momentum due to the relative motion of the electron with respect to the observer. This latter part must be

$$P_{\mathrm{II}}{}^\mu = -\frac{1}{c}\,\frac{4}{3}\,\frac{Z^2}{2\,\varrho}\,\{u^\mu - (u\,\hat{u})\,\hat{u}^\mu\} , \qquad (IV, 2)$$

in order that $(II, 14)$ remains valid:

$$P^\mu = P_{\mathrm{I}}{}^\mu + P_{\mathrm{II}}{}^\mu = \frac{1}{c}\,\frac{Z^2}{2\,\varrho}\,\left\{\tfrac{4}{3}(u\,\hat{u})\,\hat{u}^\mu - \tfrac{1}{3}\,u^\mu\right\} . \quad (IV, 3)$$

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* This assumption is only of heuristic character; the following manipulations are exact even for arbitrary difference between $\{u^\lambda\}$ and $\{\hat{u}^\lambda\}$.

Let us study this four-momentum of relative motion $\{P_{\mathrm{II}}{}^\mu\}$ in greater detail. If it is really due to the relative motion between electron and observer, it should have the form "mass times relative four-velocity". Indeed, with the observation $\varrho = (\tilde{z} - \hat{z})\,u = \varDelta s\,(u\,\hat{u})$ we can reformulate it as

$$P_{\mathrm{II}}{}^\mu = -\frac{1}{c}\,\frac{4}{3}\,\frac{Z^2}{2\,\varDelta s}\,\frac{u^\mu - (u\,\hat{u})\,\hat{u}^\mu}{(u\,\hat{u})} , \qquad (IV, 4)$$

and the vector $\{w^\lambda\}$, defined by

$$w^\lambda = -\,[u^\lambda - (u\,\hat{u})\,\hat{u}^\lambda]/(u\,\hat{u}) , \qquad (IV, 5)$$

has the properties of a relative-velocity vector: It is spacelike, because of

$$(w^\lambda\,\hat{u}_\lambda) = 0 ; \qquad (IV, 6)$$

in the rest system of the observer $(\{u^\lambda\} \Rightarrow \{\underset{\sim}{u}{}^\lambda\} = \{1; 0, 0, 0\})$ it has components

$$w^{\mathrm{k}} \Rightarrow \underset{\sim}{w}{}^{\mathrm{k}} = \hat{\underset{\sim}{u}}{}^{\mathrm{k}} = \frac{\boldsymbol{v}_{\mathrm{rel}}/c}{\sqrt{1 - (\boldsymbol{v}_{\mathrm{rel}}/c)^2}} , \qquad (IV, 7)$$

which is exactly the velocity of the electron as seen from the point of view of the observer; and finally its square gives the absolute value of relative velocity in invariant form, which was derived recently by Rohrlich and Aurilia [12]

$$-(w^\lambda\,w_\lambda) = 1 - 1/(u\,\hat{u})^2 = |\boldsymbol{v}_{\mathrm{rel}}/c|^2 . \quad (IV, 8)$$

Hence, as far as the relativistic kinematics of the internal relative motion is concerned, we can be satisfied with the splitting $(IV; 1, 2)$. But if we look at the pre-factor of the relative velocity $\{w^\lambda\}$ in $(IV, 4)$, we see that the notorious factor $4/3$ arises in front of the otherwise correct mass $(II, 6)$ of the particle.

This is not astonishing, because we know that this factor arises whenever the hyperplane of integration [here (σ_n)] is not parallel to the orthogonal hyperplane [here $(\sigma_\perp)$] in that point of the world line, where the signal (here $l_{(\hat{z})}$) defining the electron's surface is emitted. This is exactly the same faulty mechanism, which leads to the $4/3$-problem in the Abraham and Lorentz model (see Section II).

V. The 4/3-Difficulty in Connection with the Schott Term

So far, we have only delt with a uniformly moving electron $(\{\hat{u}^\lambda\} = \mathrm{const})$. If one keeps to this restriction, Rohrlich's way $(I, 4)$ out of the $4/3$-

problem seems to be meaningful. But now we want to generalize Rohrlich's method to an arbitrarily accelerated particle, and we shall show that in this case the 4/3-problem is not really solved by the Rohrlich method but only transferred to the Schott term.

In order to calculate the bound four-momentum for an accelerated electron, we can proceed exactly as in Sect. II and we will just find the result (II, 14), where $\{u^\lambda\}$ is now the four-velocity in $\{\bar{z}^\lambda\} = \{z^\lambda_{(s)}\}$ and $\{\hat{u}^\lambda\}$ the four-velocity in $\{\hat{z}^\lambda\} = \{z^\lambda_{(s-\Delta s)}\}$. Because of the applicability of Gauß' integral theorem, only two points on the world line enter the expression (II, 14), even in this quite general case of an arbitrarily curved world line (see Reference [8]). The only difference to the case of uniform motion is, that we can no longer simplify $\varrho = (z - \hat{z}) u = \Delta s (u \hat{u})$ but have now to leave $\varrho = (z - \hat{z}) u$. But this is only an indication that we have correctly delt with the retardation effects (cf. Figure 2).
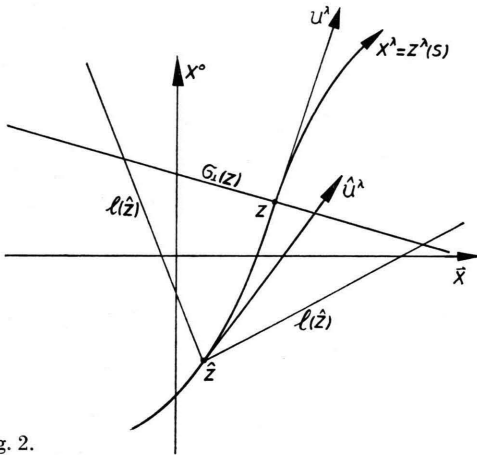

Fig. 2.

In order to recover the Schott term now, we make the usual transition to a point particle, $\Delta s \to 0$. The point $\{z^\lambda_{(s)}\}$ is held fixed in this process and $\{\hat{z}^\lambda\}$ moves towards $\{z^\lambda_{(s)}\}$. The sphere of radius ϱ in $(\sigma_\perp)$, which has been cut out for the sake of finiteness of the integration, shrinks into point $\{z^\lambda_{(s)}\}$ and therefore $\{P_I^\mu\}$ diverges $(\varrho \to \Delta s)$:

$$P_I^\mu \to \frac{1}{c} \frac{Z^2}{2 \Delta s} u^\mu. \qquad (V, 1)$$

One can think this divergence problem be solved by mass-renormalization, which is however not our own point of view (see References [10, 11]). But *the*

4/3-problem persists in the Schott term, as can be seen from formula (IV, 2) with $\hat{u}^\lambda = u^\lambda - \Delta s \, \dot{u}^\lambda \ldots$

$$P_{II}^\mu = -\frac{1}{c} \frac{Z^2}{2 \Delta s} \frac{4}{3} \{u^\mu - (u \, \hat{u}) \, \hat{u}^\mu\}$$

$$\Rightarrow -\frac{1}{c} \frac{4}{3} \frac{Z^2}{2 \Delta s} (\Delta s \, \dot{u}^\mu + \ldots) \qquad (V, 2)$$

$$\Rightarrow -\frac{1}{c} \frac{2}{3} Z^2 \dot{u}^\mu.$$

We take this as a strong objection to the Lorentz-Dirac equation of motion

$$m \, c^2 \, \dot{u}^\lambda = K^\lambda + \tfrac{2}{3} Z^2 \{\ddot{u}^\lambda + (\dot{u} \, \dot{u}) u^\lambda\}, \qquad (V, 3)$$

which incorporates the Schott term (second member of the right-hand side) in an indispensable way. The present point of criticism of the Lorentz-Dirac theory arises here from a physical interpretation of the Schott term: This term is the energy-momentum of the relative motion of the bound velocity fields (Coulomb field) with respect to the plane $(\sigma_{\perp(z)})$. Since the fields on $(\sigma_{\perp(z)})$, which define instantaneously the electron's four-momentum in $\{z^\lambda\}$ according to the Rohrlich method, have been emitted at earlier times, they carry an energy-momentum density, which belongs to these earlier states of motion. The excess over the contribution of the rigid Coulomb field, moving with the instantaneous velocity $\{u^\lambda\}$, is P_{II}^μ, or in the point limit: the Schott term according to (V, 2). This relative four-momentum is however contaminated with the notorious "4/3-illness", as can be seen from Equation (V, 2).

It is especially instructive, to reflect on the result, which would have been discovered, if the 4/3-anomaly would be absent. Replace the factor 4/3 in (IV, 2) by unity and find

$$P_{II}^\mu \Rightarrow -\frac{1}{c} \frac{Z^2}{2 \varrho} \{u^\mu - (u \, \hat{u}) \, \hat{u}^\lambda\}. \qquad (IV, 2')$$

Hence

$$P^\mu = P_I^\mu + P_{II}^\mu \Rightarrow \frac{1}{c} \frac{Z^2}{2 \varrho} (u \, \hat{u}) \, \hat{u}^\mu,$$

or in the case of uniform motion $[\varrho = \Delta s (u \, \hat{u})]$

$$P^\mu \Rightarrow \frac{1}{c} \frac{Z^2}{2 \Delta s} \hat{u}^\mu. \qquad (V, 4)$$

This means that with the replacement $4/3 \to 1$ one can use a completely arbitrary hyperplane and will always find the result (V, 4). Generalizing this result to a curved world line would yield

$$P^\mu_{(s)} = \frac{1}{c} \frac{Z^2}{2 \Delta s} u^\mu_{(s - \Delta s)}. \qquad (V, 5)$$

An expression of this form was also suggested by the recently proposed non-local generalization [11] of the Lorentz-Dirac equation *.

We see that the existence of the factor 4/3 is an expression of the fact, that Rohrlich's integral (I, 4) over the orthogonal plane $\sigma_{\perp(z)}$ with normal $\{u^\lambda_{(s)}\}$ is not identical with the integration over that hyperplane ($\hat{\sigma}_\perp$, say), which intersects the (curved) world line also in $\{z^\lambda_{(s)}\}$ but has normal $\{u^\lambda\} = \{u^\lambda_{(s-\Delta s)}\}$ instead of $\{u^\lambda_{(s)}\}$. Clearly, the latter integration would have yield immediately (V, 5) instead of (II, 14). Now, we know that both integrations would be identical in the limit $\Delta s \to 0$, if the conservation law $T^{\mu\nu}_{/\nu} \equiv 0$ were valid everywhere. Hence, putting the factor 4/3 equal to unity appears as an artificial trick ** to simulate the vanishing of the tensor divergence ($T^{\mu\nu}_{/\nu} \equiv 0$). We can test this point of view by rewriting the difference (III, 6) of the two hypersurface integrations for the present purpose as

$$\int T^\mu_{\ \nu} \, \mathrm{d}^3 \hat{\sigma}_\perp^{\ \nu} - \int T^{\mu\nu} \, \mathrm{d}^3 \sigma_{\perp\nu} = \int_{\Delta^3 l} T^\mu_{\ \nu} \, \mathrm{d}^3 l'^\nu_{(\hat{z})} \quad (V, 6)$$

* Observe however that we have put $m c^2 = \tfrac{4}{3} Z^2/2 \, \Delta s$ in the new theory. This indicates that an eventually possible derivation of the new non-local theory should not make use of the orthogonal hyperplanes at all. Moreover, in an expression like (V, 4) one can eliminate the Schott term completely by expanding about $\{\hat{z}^\lambda\}$ instead of $\{z^\lambda\}$; see Ref. [11].

** Consult the book of Sexl and Urbantke [1] for the superfluous 1/3!

where the right-hand side can be taken over from (III, 4) and splitted up as in (IV; 1, 2)

$$\int_{\Delta^3 l} T^\mu_{\ \nu} \, \mathrm{d}^3 l'^\nu_{(\hat{z})} = \frac{Z^2}{2 \, \Delta s} \hat{u}^\mu - \frac{Z^2}{2 \varrho} \{\tfrac{4}{3} \hat{u}^\mu (u \, \hat{u}) - \tfrac{1}{3} u^\mu\}$$
$$= \frac{Z^2}{2 \, \Delta s} \hat{u}^\mu - \frac{Z^2}{2 \varrho} u^\mu + \frac{4}{3} \frac{Z^2}{2 \varrho} \{u^\mu - (u \, \hat{u}) \hat{u}^\mu\} . \quad (V, 7)$$

Now put $4/3 \to 1$ in the last term and find with the subsidiary assumption $\varrho \to \Delta s (u \, \hat{u})$, which is however allowed strictly in the limit $\Delta s \to 0$ for curved world lines

$$\int_{\Delta^3 l} T^\mu_{\ \nu} \, \mathrm{d}^3 l'^\nu_{(\hat{z})} \to 0 .$$

So we see once more, that the substitution $4/3 \to 1$ makes the difference between the two hyperplane integrals vanish in the limit $\Delta s \to 0$ for a curved world line, as would be the case automatically for $T^{\mu\nu}_{/\nu} \equiv 0$ in all space-time.

These considerations in turn suggest that the Schott term, being intimately connected with the 4/3-problem, will not emerge in a consistent theory, in which the 4/3-problem is not existent from the very beginning.

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